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Coloring by two-way independent sets

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ABSTRACT

The paper stems from an attempt to investigate a somewhat mysterious phenomenon: conditions which suffice for the existence of a "large" set satisfying certain conditions (e.g., a large independent set in a graph) often suffice (or at least are conjectured to suffice) for the existence of a covering of the ground set by few sets satisfying these conditions (in the example of independent sets in a graph this means that the graph has small chromatic number). We consider two conjectures of this type, on coloring by sets which are "two-way independent", in the sense of belonging to a matroid and at the same time being independent in a graph sharing its ground set with the matroid. We prove these conjectures for matroids of rank 2. We also consider dual conjectures, on packing bases of a matroid, which are independent in a given graph.

1. Introduction

It was probably Edmonds who first realized that many concepts in matching theory can be described in terms of two structures imposed on the same ground set. For example, a matching in a bipartite graph is a set of edges belonging to two partition matroids, induced on the edge set of the graph by the incidence relations on the two sides. Edmonds' two matroids intersection theorem [8] provides for the existence of a large set belonging to two matroids on the same ground set. Besides two matroids, there are other types of pairs of structures on the same set that are of interest. In [13,2], for example, conditions are studied for the existence of a large set which belongs to a matroid and is independent in a graph, where the matroid and the graph share the same vertex set. In [1] an even more general setting is studied, in which a matroid is combined with a general simplicial complex (namely, a closed down hypergraph). In all these cases we shall name (for the purposes of this paper) a set belonging to the two given structures a "two-way independent set".

A rather mysterious phenomenon appears in many such settings (and also in other parts of matching theory). It is that conditions guaranteeing the existence of large two-way independent sets often suffice, or are at least suspected to suffice, for the existence of an abundance of such sets, in the sense that there is an "economical" (that is, small) covering of the ground set by such sets. Here is a partial list:

(1) A straightforward application of Hall's theorem yields that in a regular bipartite graph there exists a perfect matching. This, in turn, implies that in fact there exists a partition into perfect matchings.
(2) The Hajnal–Szemerédi theorem: if all degrees in a given graph G are smaller than k, then by a greedy argument there exists an independent set of size $\frac{|V(G)|}{k}$ (assume, for convenience, that k divides $|V(G)|$). Such a set can be viewed as a two-way independent set, where the two structures are the independence complex of the graph and the $\frac{|V(G)|}{k}$-uniform matroid. The Hajnal–Szemerédi theorem [12] states that in fact there exists a cover by k such sets. (See [17] for a relatively short proof.)

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(3) Let H be a 3-partite hypergraph, whose sides V_1, V_2 and V_3, are of sizes n, $2n$ and $2n$, respectively. Suppose that every pair (u, v), where $u \in V_1$ and $v \in V_2 \cup V_3$, is contained in precisely one edge of H, and every pair $(u, v) \in V_2 \times V_3$ is contained in at most one edge. It is not hard to prove then that there exists in H a matching of size n. This can be formulated as: "Every Latin $n \times 2n$ rectangle has a transversal of size n." (In [4] something stronger was shown: it suffices to assume that $|V_1| = n$, $|V_2| \geq 2n - 1$, every pair (u, v), where $u \in V_1$ and $v \in V_2$, is contained in precisely one edge of H and every pair (u, v) for which $u \in V_1$ and $v \in V_3$ is contained in at most one edge of H.) The "coloring" version of this fact was conjectured by Hilton [16], who suggested that under the same conditions there exists a partition of the edge set of the hypergraph into $2n$ matchings. Häggkvist and Johanssen [11] noted that Hilton's conjecture is true if $2n$ is replaced by $4n$, and proved an asymptotic version of the conjecture.

(4) By a theorem of Füredi [9], in an r-uniform hypergraph H there exists a matching of size $(r - 1 + \frac{1}{r})\frac{|E(H)|}{D(H)}$ (here, and below, $D(H)$ is the maximal degree of a vertex in the hypergraph H). A conjecture of Füredi, Kahn and Seymour [10] is that $\chi'(H) \leq (r - 1 + \frac{1}{r})D(H)$, namely there exists a decomposition of $E(H)$ into matchings of average size at least $(r - 1 + \frac{1}{r})\frac{|E(H)|}{D(H)}$. In fact, possibly there exists a decomposition into matchings each of which has precisely this size, rounded up or down.

(5) An analogous situation exists in r-partite hypergraphs, where $r - 1 + \frac{1}{r}$ is replaced by $r - 1$.

To state yet another conjecture of the same flavor, we shall need some notation.

Definition 1.1. Given a simplicial complex $\mathcal{C}$, a $\mathcal{C}$-*coloring* (or a "coloring by $\mathcal{C}$") is a cover of the ground set by simplices (namely, edges) from $\mathcal{C}$. The *chromatic number* $\chi(\mathcal{C})$ of $\mathcal{C}$ is the minimal size of a $\mathcal{C}$-coloring, namely the minimal number of simplices in $\mathcal{C}$ whose union is V.

The complex of independent sets of vertices in a graph G is denoted by $\mathcal{I}(G)$. Thus in this notation $\chi(\mathcal{I}(G))$ is simply the chromatic number of G.

For the following definition, and also for some results cited below, we shall consider only matroids without loops. That is, in all matroids we shall consider every singleton set is independent. As usual, $\rho_{\mathcal{M}}(X)$ will denote $\max\{|I| : I \subset X,\ I \in \mathcal{M}\}$, and $sp_{\mathcal{M}}(X)$ denotes the span of X, namely X together with those elements v of $V \setminus X$ such that $I \cup \{v\} \notin \mathcal{M}$ for some $I \subset X,\ I \in \mathcal{M}$.

Definition 1.2. Given a matroid $\mathcal{M}$, we write $\Delta(\mathcal{M})$ for the maximum of $\frac{|X|}{\rho_{\mathcal{M}}(X)}$ over all nonempty subsets X of $V(\mathcal{M})$, which is the same as $\max\{\frac{|sp_{\mathcal{M}}(A)|}{|A|} : A \subseteq V(\mathcal{M}), A \neq \emptyset\}$.

Edmonds [7] proved:

Theorem 1.3. $\chi(\mathcal{M}) = \lceil \Delta(\mathcal{M}) \rceil$.

In fact, it is not hard to show that $\Delta(\mathcal{M})$ is equal to the fractional covering number of the matroid, $\chi^*(\mathcal{M})$, which is real valued relaxation of the chromatic number.

Edmonds' two matroids intersection theorem implies:

Theorem 1.4. *If $\mathcal{M}$, $\mathcal{N}$ are two matroids on the same ground set V, then*

$$\max\{|I| : I \in \mathcal{M} \cap \mathcal{N}\} \geq \frac{|V|}{\max(\chi(\mathcal{M}), \chi(\mathcal{N}))}.$$

In [1] it was conjectured that this theorem "almost" characterizes the minimal number of two-way independent sets needed to cover the ground set:

Conjecture 1.5. $\chi(\mathcal{M} \cap \mathcal{N}) \leq \max(\chi(\mathcal{M}), \chi(\mathcal{N})) + 1$.

It is easy to prove the fractional version of this conjecture (see e.g. [1]), which implies that there are "many" two-way independent sets of size at least $\frac{|V|}{\max(\chi^*(\mathcal{M}), \chi^*(\mathcal{N}))}$ (note that $\frac{|V|}{\max(\chi^*(\mathcal{M}), \chi^*(\mathcal{N}))} \geq \frac{|V|}{\max(\chi(\mathcal{M}), \chi(\mathcal{N}))}$):

Theorem 1.6. $\chi^*(\mathcal{M} \cap \mathcal{N}) = \max(\chi^*(\mathcal{M}), \chi^*(\mathcal{N}))$.

In [1] a weakened version of Conjecture 1.5 was proved:

Theorem 1.7. *For any two matroids $\mathcal{M}$ and $\mathcal{N}$ on the same ground set there holds* $\chi(\mathcal{M} \cap \mathcal{N}) \leq 2\max(\chi(\mathcal{M}), \chi(\mathcal{N}))$.

2. The coloring number of the intersection of a matroid and a simplicial complex

The conjecture addressed in the present paper concerns another (apparent) phenomenon of the same family. In order to formulate it, we shall need a few more concepts on matroids and complexes.

For a complex $\mathcal{C}$ we denote by $\eta(\mathcal{C})$ the topological connectivity of $\mathcal{C}$, plus 2. It is not really necessary to know what this means, since we shall just quote certain properties of η. But for the reader who wishes to have a general understanding of what it means, a simple definition is that $\eta(\mathcal{C})$ is the smallest dimension of a hole in $\|\mathcal{C}\|$, the geometric realization of $\mathcal{C}$. For example, if $\mathcal{C}$ is nonempty, but not path-connected, then $\eta(\mathcal{C}) = 1$ since there is a hole of dimension 1—two points that cannot be connected by a path. If $\mathcal{C}$ is a connected graph, then $\eta(\mathcal{C}) = \infty$ if $\mathcal{C}$ is a tree, since then there is no hole at all, and $\eta(\mathcal{C}) = 2$ if $\mathcal{C}$ contains a cycle, since this cycle cannot be filled by simplices from $\mathcal{C}$ (there being no simplices of dimension 2 at all in $\mathcal{C}$,) and thus this cycle is a hole of dimension 2. Another example: if $\mathcal{C}$ is the set of all subsets of size 3 or less of a set of size 4, then its geometric realization is the boundary of a tetrahedron. This boundary forms a hole of dimension 3, which has no filling in $\mathcal{C}$, and thus $\eta(\mathcal{C}) = 3$.

Recall that a graph is called *chordal* if every cycle of length larger than 3 in it has a chord. A graph is *stably wide* if every induced subgraph H of G contains an independent set requiring $\gamma(H)$ (the domination number of H) vertices to dominate. The simplest examples of stably wide graphs are chordal graphs, and cycles of length divisible by 3 (see [2]).

We shall need the following facts about η:

Theorem 2.1. (1) *For a matroid $\mathcal{M}$ with no loops there holds $\eta(\mathcal{M}) = \rho(\mathcal{M})$, unless there exists a vertex belonging to all bases, in which case $\eta(\mathcal{M}) = \infty$.*

(2) *For a graph G there holds $\eta(\mathcal{I}(G)) \geq \frac{\tilde{\gamma}(G)}{2}$, where $\tilde{\gamma}(G)$ denotes the* total domination number *of G, namely the minimal number of vertices needed to dominate the graph, when a vertex does not dominate itself. (This can be found in [19], or [5].)*

(3) [6] *If G is stably wide, then $\eta(\mathcal{I}(G)) \geq \gamma(G)$.*

We write $\bar{\eta}(\mathcal{C})$ for the minimum of $\eta(\mathcal{C})$ and the largest size of a simplex in $\mathcal{C}$. Namely, $\bar{\eta}(\mathcal{C}) = \eta(\mathcal{C})$, unless $\eta(\mathcal{C}) = \infty$, in which case $\bar{\eta}(\mathcal{C})$ is the maximal size of a simplex.

Extending the definition of Δ from matroids to general complexes, we define $\Delta(\mathcal{C})$ for a complex $\mathcal{C}$ as the maximum, over all nonempty subsets A of $V(\mathcal{C})$, of $\frac{|A|}{\bar{\eta}(\mathcal{C}|A)}$.

In [1] the following result was proved (Theorem 8.2):

Theorem 2.2. *If $\mathcal{M}$ and $\mathcal{C}$ are a matroid and a complex on V then* $\max\{|I| : I \in \mathcal{M} \cap \mathcal{C}\} \geq \frac{|V|}{\max(\Delta(\mathcal{M}), \Delta(\mathcal{C}))}$.

Since clearly in a graph G there holds $\tilde{\gamma}(G) \geq \frac{|V|}{D(G)}$ and $\gamma(G) \geq \frac{|V|}{D(G)+1}$, by Theorems 2.1 and 2.2 and the fact that $\chi(\mathcal{M}) = \lceil \Delta(\mathcal{M}) \rceil$ we have:

Corollary 2.3. *If $\mathcal{M}$ and G are a matroid and a graph, respectively, on V, then*

$$\max\{|I| : I \in \mathcal{M} \cap \mathcal{I}(G)\} \geq \frac{|V|}{\max(\chi(\mathcal{M}), 2D(G))}.$$

If G is chordal then

$$\max\{|I| : I \in \mathcal{M} \cap \mathcal{I}(G)\} \geq \frac{|V|}{\max(\chi(\mathcal{M}), D(G) + 1)}.$$

It has been conjectured (see, e.g., [14,15,3]) that the "matchability implies colorability" phenomenon holds also for the first of these inequalities, if the matroid is a partition matroid. In fact, we propose that this is true for both inequalities, if the matroid is a "truncated partition matroid", which means a matroid obtained by intersecting a partition matroid with a k-uniform matroid for some k. Namely:

Conjecture 2.4. *If $\mathcal{M}$ is a truncation of a partition matroid and G is a graph sharing with $\mathcal{M}$ its ground set, then:*

$$\chi(\mathcal{M} \cap \mathcal{I}(G)) \leq \max(\chi(\mathcal{M}), 2D(G)).$$

Conjecture 2.5. *If $\mathcal{M}$ is a truncation of a partition matroid and G is chordal then:*

$$\chi(\mathcal{M} \cap \mathcal{I}(G)) \leq \max(\chi(\mathcal{M}), D(G) + 1).$$

In the case of general matroids, both conjectures are refuted by the same well-known example ([20], Section 42.6c): $\mathcal{M}$ is the graphic matroid on $E(K_4)$, and the graph G is a matching of size 3, in which every edge of K_4 is connected to the one edge it does not meet. It is easy to see that $\chi(\mathcal{M}) = 2D(G) = D(G) + 1 = 2$, while $\chi(\mathcal{M} \cap \mathcal{I}(G)) = 3$.

Haxell [14] showed that $\chi(\mathcal{M} \cap \mathcal{I}(G)) \leq \max(\chi(\mathcal{M}), 3D(G) - 1)$ if $\mathcal{M}$ is a partition matroid. In [15] it was shown that for large enough $D(G)$, the factor 3 can be replaced by 2.75. In [2] the idea of [14] was used to show that if G is chordal and $\mathcal{M}$ is a partition matroid then $\chi(\mathcal{M} \cap \mathcal{I}(G)) \leq \max(\chi(\mathcal{M}), 2D(G) + 1)$. The fractional version of Conjecture 2.4 was proved in [1]. The fractional version of Conjecture 2.5 remains open.

3. Proof of Conjecture 2.4 for matroids of rank 2

In a matroid of rank 2 the independent sets form a graph, and hence we have at our disposal matchability tools from graph theory, in particular Tutte's theorem. Hence it is possible to handle the two conjectures in this case. In this section we prove the first conjecture:

Theorem 3.1. *For a matroid $\mathcal{M}$ of rank 2 Conjecture 2.4 is true, namely for any graph G there holds $\chi(\mathcal{M} \cap \mathcal{I}(G)) \leq \max(\chi(\mathcal{M}), 2D(G))$.*

The proof will require some more graph theoretical definitions and notation. We write $|G|$ for the number of vertices in a graph G, namely $|G|$ stands for $|V(G)|$. For a graph G we denote by $\bar{G}$ the complement graph of G. For a set A of vertices in a graph G denote by $N_G(A)$ the closed neighborhood of A, namely the set of vertices that either belong to A or are adjacent to some vertex of A. We use the same notation, $N_G(a)$ for the closed neighborhood of a vertex a. If the identity of G is clear, we shall omit its mention and write $N(A)$. The degree of a vertex in a graph G is denoted by $d_G(v)$ (if the identity of G is clear from the context the subscript is omitted). The maximal degree of a vertex in a hypergraph H (in particular, a graph) is denoted by $D(H)$ (we use this rather than the more common notation $\Delta(H)$ since Δ has been used already with another meaning). The minimal degree of a vertex in H is denoted by $d(H)$. The maximal size of a matching in a graph G is denoted by $\nu(G)$. Given a set F of edges in a graph we write $supp(F)$ for the union of all edges in F. As usual, given a set T in a graph G we denote by $G[T]$ the graph induced by G on T, and by $G - T$ the graph $G[V(G) \setminus T]$. The set of odd components and the set of even components of a graph G are denoted by $\mathcal{O}(G)$ and $\mathcal{E}(G)$ respectively.

Lemma 3.2. *Let G be bipartite graph with parts U and W, and assume that $|U| \geq |W|$. If $d(u) \geq |W| - \frac{|U|}{2}$ for all $u \in U$ and $d(w) \geq \frac{|U|}{2}$ for all $w \in W$, then $\nu(G) = |W|$.*

Proof. We shall show that every matching F of size less than $|W|$ can be augmented. Choose vertices $u \in U \setminus supp(F)$ and $w \in W \setminus supp(F)$. By the assumption of the lemma, $d(u) + d(w) \geq |W| > |F|$. But this implies that either there exists a vertex not belonging to $supp(F)$ connected to u or to w, or there exists an edge $(a, b) \in F$ such that (u, b) and (a, w) are both edges of the graph. In both cases F can be augmented, in the first by the addition of an edge, in the second by replacing (a, b) by (u, b) and (a, w). $\square$

Lemma 3.3. *Let $G = (V, E)$ be a t-partite graph, where $t \geq 2$, and assume that $|V|$ is even. Assume furthermore that the parts S_i, $i \leq t$, of G satisfy: $|S_i| \leq \frac{|V|}{2}$, and that for every vertex $x \in S_i$ there holds $d(x) \geq \frac{3}{4}|V| - |S_i|$. Then G has a perfect matching.*

Proof. The case $t = 2$ of the lemma follows from Lemma 3.2. The case $t > 3$ can be reduced to the case $t = 3$, by unifying the two smallest parts into one part, and deleting all edges between these parts from the graph. This does not violate the condition that all parts are of size at most $\frac{|V|}{2}$, or the condition on the degrees of vertices in each part. Thus we henceforth assume that $t = 3$.

Assume that G does not have a perfect matching. By Tutte's theorem [21], there exists a set T such that $|\mathcal{O}(G-T)| > |T|$. Since $|V|$ is even, this implies $|\mathcal{O}(G - T)| > |T| + 1$. Write $\mathcal{O} = \mathcal{O}(G - T)$, $\mathcal{E} = \mathcal{E}(G - T)$. By a well-known strengthening of Tutte's theorem, the Gallai–Edmonds structure theorem (see e.g.[18], 3.2.1), we may assume that every graph in $\mathcal{E}$ has a perfect matching, and every graph in $\mathcal{O}$ is factor-critical (namely, the deletion of any vertex from it results in a graph having a perfect matching). Let $T_i = T \cap S_i$ for $i \leq 3$. Since $|T| + 1 < |\mathcal{O}|$, we have:

$$|V| = |T| + \sum_{O\in\mathcal{O}} |O| + \sum_{H\in\mathcal{E}} |H| > 2|T| + 1 + \sum_{O\in\mathcal{O}} (|O| - 1) + \sum_{H\in\mathcal{E}} |H|. \tag{1}$$

Case 1: $T_i = S_i$ for some i. Without loss of generality we can assume that $T_3 = S_3$. Since $|T| < \frac{|V|}{2}$, we then have $T_i \neq S_i$ for $i = 1, 2$. Since a factor-critical graph cannot be bipartite, all graphs in $\mathcal{O}$ are singletons, namely isolated vertices in $G - T$. Every graph $H \in \mathcal{E}$ is bipartite and has a perfect matching, and hence $|V(H) \cap S_i| = \frac{|H|}{2}$ for every such graph H and every $i = 1, 2$. Hence, if all isolated vertices of $G-T$ lie in the same part, say S_1, then $|S_1| > |S_2|+|S_3|$, contradicting the assumption that $|S_1| \leq \frac{|V|}{2}$. Thus we can assume that there exist $x \in S_1$ and $y \in S_2$ which are isolated in $G - T$. Then, $|T_2| + |T_3| \geq d(x)$ and $|T_1| + |T_3| \geq d(y)$. Thus $|T| + |T_3| \geq d(x) + d(y) \geq (\frac{3|V|}{4} - |S_1|) + (\frac{3|V|}{4} - |S_2|) = \frac{3|V|}{2} - |V| + |S_3| = \frac{|V|}{2} + |S_3|$, implying that $|T| \geq \frac{|V|}{2}$. But since $|\mathcal{O}| > |V|$, we have $|V| > 2|T|$.

Case 2: $T_i \neq S_i$ for $i = 1, 2, 3$. Since, as noted, factor-critical graph $O \in \mathcal{O}$ is not bipartite, either for all i we have $V(O) \cap S_i \neq \emptyset$ (in this case $|V(O) \cap S_i| \leq \lfloor\frac{|O|}{2}\rfloor$) or O is a singleton. Since every even component $E \in \mathcal{E}$ has a perfect matching, for all i it follows that $|V(E) \cap S_i| \leq \frac{|E|}{2}$.

Suppose that there exist three components K_1, K_2, K_3 of $G - T$ (even or odd), from which it is possible to choose three vertices $x_i \in K_i \cap (S_i \setminus T_i)$. For each $i = 1, 2, 3$ we have $|T| - |T_i| + |K_i| - 1 \geq d(x_i) \geq \frac{3}{4}|V| - |S_i|$, and summing up these inequalities we get $2|T| + \sum_{i=1}^{3}(|K_i| - 1) \geq \frac{5}{4}|V|$, which contradicts Eq. (1).

Thus we may assume that there is no such choice of vertices. Construct a bipartite graph Γ one side of which consists of the connected components of $G - T$ and the other is the sets S_i, and in which an edge exists between a component K and a set S_i if $V(K) \cap S_i \neq \emptyset$. By the above, $\nu(\Gamma) < 3$. By König's theorem it follows then that there are two possibilities:

(i) $T = \emptyset$, there are no even components, and there are just two odd components O_1 and O_2. If one of them is an isolated vertex x, then $d(x) = 0$, violating the condition of the lemma. Thus we may assume that O_1 and O_2 both meet all S_i's. Assume that S_1 is the smallest of the sets S_i. Since we are assuming that all sets S_i are nonempty, this implies that $|S_1| < \frac{|V|}{2}$. Select $x_1 \in S_1 \cap O_1$ and $x_2 \in S_1 \cap O_2$. Then $\frac{3}{4}|V| - |S_1| \le d(x_1) \le |V(O_i) \setminus S_1|$ for $i = 1, 2$, implying $|V(O_i) \setminus S_1| \le \frac{|V|}{4}$. Adding these inequalities, we get $\frac{|V|}{2} \le |S_1|$, a contradiction.

(ii) $G - T$ consists of isolated vertices, all contained in one set (say) S_1, together with one more component K. Then $(S_2 \cup S_3) \setminus T \subseteq V(K)$. Write $K_1 = V(K) \cap S_1$. Select an isolated vertex $x_1 \in S_1 \setminus T_1$ and a vertex $x_2 \in V(K) \cap S_2$. The condition $|\mathcal{O}| > |T|$ then reads:

$$|T| < |S_1| - |K_1| - |T_1|. \tag{2}$$

The degree condition on x_1, together with the fact that x_1 is a singleton component in $G - T$ yields:

$$\frac{3}{4}|V| - |S_1| \le d(x_1) \le |T_2| + |T_3|. \tag{3}$$

Similarly, the degree condition on x_2 yields:

$$\frac{3}{4}|V| - |S_2| \le d(x_2) \le |S_3| + |K_1| + |T_1|. \tag{4}$$

Adding up (2) and (3) and dividing by 2 yields:

$$\frac{3}{8}|V| + |T_1| + \frac{1}{2}|K_1| < |S_1|. \tag{5}$$

Adding (4) and (5) and simplifying yields:

$$\frac{1}{4}|V| < |K_1|. \tag{6}$$

But by (5) this yields $|S_1| > \frac{1}{2}|V|$, a contradiction. □

We can now complete the proof of Theorem 3.1. As already noted, we are assuming that every singleton belongs to $\mathcal{M}$. This implies that the relation defined by $x \sim y$ if $\{x, y\} \notin \mathcal{M}$ is an equivalence relation. Let $S_1, S_2, \dots, S_t$ be the equivalence classes of this relation (these are called the "parallelism classes" of $\mathcal{M}$). Since $\mathcal{M}$ has rank 2, the bases of M are precisely those sets which consist of two points from different classes S_i. By definition, we have $\Delta(\mathcal{M}) = \max(\max|S_i|, \frac{|V|}{2})$. If $\Delta(\mathcal{M}) = \frac{|V|}{2}$ is not an integer, we can add an element to some set S_i and have it isolated in G, thus not changing $\chi(\mathcal{M})$ and making $\chi(\mathcal{M}) = \Delta(\mathcal{M})$. Hence we can assume that $\chi(\mathcal{M}) = \Delta(\mathcal{M})$. If $\Delta(\mathcal{M}) < 2D(G)$, we can add to some sets S_i new elements which are isolated in G, thus not changing $D(G)$ and obtaining a matroid $\mathcal{M}'$ with $\Delta = 2D(G)$. Thus we may assume that $\Delta(\mathcal{M}) \ge 2D(G)$.

Assume first that $\Delta(\mathcal{M}) = \max(S_i)$. Without loss of generality $\max|S_i| = |S_1|$. Then $|S_1| \ge |V \setminus S_1|$. Let J be obtained from $\bar{G} \cap \mathcal{M}$ by deleting all edges between vertices of S_i and S_j for $i \ne j \in \{2, \dots, t\}$. Then J is a bipartite graph with sides S_1 and $V \setminus S_1$. For $x \in S_1$, $d_J(x) \ge |V \setminus S_1| - D(G) \ge |V \setminus S_1| - \frac{|S_1|}{2}$ and for $y \in V \setminus S_1$, $d_J(y) \ge |S_1| - D(G) \ge \frac{|S_1|}{2}$. By Lemma 3.2, $\nu(J) = |V \setminus S_1|$. Let F be a matching of size $|V \setminus S_1|$ and let $A = S_1 \setminus supp(F)$. Then $F \cup \{\{v\} \mid v \in A\}$ is a partition of $V(G)$ into sets belonging to $\mathcal{I}(G) \cap \mathcal{M}$ and its size is $|S_1|$, proving that $\chi(\mathcal{M} \cap I(G)) \le \Delta(\mathcal{M})$.

Assume next that $\Delta(\mathcal{M}) = \frac{|V|}{2}$. Since $\max(\max|S_i|, \frac{|V|}{2}) = \frac{|V|}{2}$, we have $|S_i| \le \frac{|V|}{2}$ for all i. Define $J = \bar{G} \cap \mathcal{M}$. Then, for all i and for all $x \in S_i$, we have $d_J(x) \ge |V| - D(G) - |S_i| = \frac{3|V|}{4} - |S_i|$. By Lemma 3.3, J has a perfect matching. Hence $\chi(\mathcal{I}(G) \cap \mathcal{M}) = \nu(\bar{G} \cap \mathcal{M}) = \frac{|V|}{2} = \Delta(\mathcal{M})$. This completes the proof of the theorem.

4. The chordal graphs case

As noted in the previous section, a matroid of rank 2 is a truncated partition matroid. Hence, the rank 2 case of Conjecture 2.5 is:

Theorem 4.1. *For a matroid $\mathcal{M}$ of rank 2 Conjecture 2.5 is true, namely if G is chordal then $\chi(\mathcal{M} \cap \mathcal{I}(G)) \le \max(\chi(\mathcal{M}), D(G) + 1)$.*

Lemma 4.2. *Let G be a graph such that $\bar{G}$ is chordal. If $V = V(G)$ is partitioned into sets V_1 and V_2 such that $|V_1| \ge |V_2|$ and $D(\bar{G}) \le |V_1| - 1$, then in the bipartite graph $B = G - (E(G[V_1]) \cup E(G[V_2]))$ the set V_2 is matchable into V_1.*

Proof. Assuming the negation of the lemma, by Hall's theorem there exists a set $X \subseteq V_2$ such that $|X| > |N_B[X]|$. Write $P = \bar{G}[X \cup (V_1 \setminus N_B[X])]$. Then P contains a complete bipartite graph on $X \times (V_1 \setminus N_B[X])$. There do not exist vertices $x_1, x_2 \in X$ and $y_1, y_2 \in V_1 \setminus N_B[X]$ such that $(x_1, x_2), (y_1, y_2) \notin E(P)$, since otherwise (x_1, y_1, x_2, y_2) would be a chordless cycle in $\bar{G}$, contradicting the chordality of $\bar{G}$. Thus at least one of the graphs $P[X]$ and $P[V_1 \setminus N_B[X]]$ is complete. Hence there exists a vertex $v \in X \cup (V_1 \setminus N_B[X])$ which is connected to all vertices of P apart from itself, implying that $d_P(v) \geq (|X| + |V_1| - |N_B[X]|) - 1 > |V_1| - 1$. This contradicts the assumption that $D(\bar{G}) \leq |V_1| - 1$. □

Lemma 4.3. *Let G be a graph such that $\bar{G}$ is chordal and $d(G) \geq \frac{|V|}{2}$. If $\{S_i\}_{i=1}^{t}$ is a partition of V such that $|S_i| \leq \frac{|V|}{2}$ for all i then the graph $L = G - (\bigcup_{i=1}^{t} E(G[S_i]))$ has a perfect matching.*

Proof. As in the proof of Lemma 3.3 we may assume that $t = 3$. By Tutte's theorem, if L does not possess a perfect matching then there exists a set T such that $|\mathcal{O}(L - T)| > |T|$. Write $\mathcal{O} = \mathcal{O}(L - T)$, $\mathcal{E} = \mathcal{E}(L - T)$. As stated above, we may assume that every graph in $\mathcal{E}$ has a perfect matching, and every graph in $\mathcal{O}$ is factor-critical. Since $|T| < |\mathcal{O}|$, we have $|V| = |T| + \sum_{O \in \mathcal{O}} |O| + \sum_{H \in \mathcal{E}} |H| > 2|T| + \sum_{O \in \mathcal{O}} (|O| - 1) + \sum_{O \in \mathcal{E}} |H|$. Hence

$$|T| + \frac{\sum\limits_{O \in \mathcal{O}} (|O| - 1)}{2} + \frac{\sum\limits_{H \in \mathcal{E}} |H|}{2} < \frac{|V|}{2}. \tag{7}$$

Let $\mathcal{K} = \mathcal{O} \cup \mathcal{E}$. We can write Eq. (7) as:

$$|T| + \sum_{K \in \mathcal{K}} \left\lfloor \frac{|K|}{2} \right\rfloor < \frac{|V|}{2}. \tag{8}$$

Since $\bar{G} - T$ is chordal, it contains a simplicial vertex, namely a vertex $v \in V(G) \setminus T$ such that $V \setminus N_{(G-T)}(v)$ is an independent set in $G - T$. Without loss of generality suppose that $v \in S_1 \setminus T$. Define T_i as $T \cap S_i$ for $i = 1, 2, 3$.

Case 1: $T_i = S_i$ for some i. Without loss of generality, suppose $T_3 = S_3$. Since a bipartite factor-critical graph is a singleton, the assumption that the graphs in $\mathcal{O}$ are factor-critical implies that all these graphs are singletons. Since any even component $H \in \mathcal{E}$ is bipartite and has a perfect matching, $|V(H) \cap S_i| = \frac{|H|}{2}$ for $i = 1, 2$.

The simplicial vertex v is either a singleton odd component, or it lies in an even component of $L - T$. Assume first that $\{v\}$ is an odd component. Then, by the simpliciality property of v, no two vertices $x, y \in S_2 \setminus T$ are connected in G. This implies that for every $u \in S_2 \setminus T$ such that $\{u\}$ is an odd component we have $N_G(u) \subseteq T$. Since $|T| < \frac{|V|}{2}$, this contradicts the assumption that $d(G) \geq \frac{|V|}{2}$. Thus there is no odd component contained in S_2. A vertex $u \in S_2$ lying in an even component H can be adjacent in G only to $T \cup (V(H) \cap S_1)$, but by Eq. (7), $|T \cup (V(H) \cap S_1)| = |T| + |H|/2 < \frac{|V|}{2}$, and again this contradicts the assumption that $d(G) \geq \frac{|V|}{2}$. Thus there is also no $H \in \mathcal{E}$ meeting S_2, meaning that $T_2 = S_2$. This contradicts the fact that $|T| < \frac{|V|}{2}$.

In the second case, v lies in an even component $H_0 \in \mathcal{E}$. Then no two vertices $x, y \in S_2 \setminus (T \cup V(H_0))$ are adjacent in G. If u is a vertex in S_2 such that $\{u\} \in \mathcal{O}$, then u is adjacent in G only to $T \cup (V(H_0) \cap S_2)$. By Eq. (7) we get a contradiction to the assumption that $d(G) \geq \frac{|V|}{2}$. This proves that no odd component $O \in \mathcal{O}$ is contained in S_2. A vertex $u \in S_2$ lying in an even component $H \neq H_0$ can be adjacent in G only to $T \cup (V(H) \cap S_1) \cup (V(H_0) \cap S_2)$, but by Eq. (7) this, too, contradicts the assumption that $d(G) \geq \frac{|V|}{2}$. Thus, the only even component in $L - T$ is H_0. So $S_1 = (\bigcup\{V(O) : O \in \mathcal{O}\}) \cup (V(H_0) \cap S_1) \cup T_1$ and $S_2 = (V(H_0) \cap S_2) \cup T_2$. We have then $|S_1| \geq |\mathcal{O}| + \frac{|H_0|}{2} > |T| + \frac{|H_0|}{2} \geq |S_2| + |S_3|$, contradicting the assumption that the size of every S_i is less than or equal to $\frac{|V|}{2}$.

Case 2: $T_i \neq S_i$ for all i. Since every odd component $O \in \mathcal{O}$ is factor-critical, and since a bipartite graph which is not a singleton cannot be factor-critical, every such O is either a singleton or it intersects every S_i. In the latter case, if $|V(O) \cap S_i| > \frac{|O|-1}{2}$ for some i, then for every $z \in V(O) \cap S_j$, $j \neq i$, the graph $O - z$ does not have a perfect matching, contradicting the factor-criticality property of O. Hence, if O intersects every S_i, we have $|V(O) \cap S_i| \leq \frac{|O|-1}{2}$ for all i. Similarly, the fact that every even component $H \in \mathcal{E}$ has a perfect matching implies that $|V(H) \cap S_i| \leq \frac{|H|}{2}$ for all i. We have thus shown that $|V(K) \cap S_i| \leq \lfloor \frac{|K|}{2} \rfloor$ for all $K \in \mathcal{K}$ and all $i = 1, 2, 3$.

As in Case 1, we consider two subcases. In the first, v is isolated in $L - T$. Then no edge exists in G between any two vertices $x, y \in V \setminus (T \cup S_1)$. Therefore, any vertex $u \in S_2 \setminus T$ is adjacent to at most $|T| + |V(K) \cap S_1|$ vertices in G, where K is the component of $L - T$ containing u. Since $|T| + |V(K) \cap S_1| \leq |T| + \lfloor \frac{|K|}{2} \rfloor$, by Eq. (8), we have $d_G(u) < \frac{|V|}{2}$, contradicting the assumption of the lemma. Thus, $S_2 \setminus T = \emptyset$, contrary to the assumption made in the current case.

Assume next that v lies in a component K which is not a singleton. For any odd component $O \neq K$ which is not a singleton, the fact that $N_{(G-T)}(v) \cap (V(O) \setminus S_1) = \emptyset$ implies that $V(O) \setminus S_1$ is independent in $G - T$. But this means that O is bipartite, and hence cannot be factor-critical. Thus every odd component, except possibly for K, is a singleton. An odd component $\{u\}$ contained in S_i $(i = 2, 3)$ can be adjacent in G only to $T \cup (V(K) \cap S_i)$. But by Eq. (8) this yields a contradiction to the assumption that $d_G(u) \geq \frac{|V|}{2}$. A vertex $u \in S_i$ $(i = 2, 3)$ lying in an even component $H \neq K$ is adjacent only to vertices in $T \cup (V(H) \cap S_1) \cup (V(K) \cap S_i)$, which by (8) contradicts the fact that $d_G(u) \geq \frac{|V|}{2}$. Thus $L - T$ consists only of singletons in S_1 plus the component K. If for some $i \in \{2, 3\}$ we have $V(K) \cap S_i = \emptyset$, then $T_i = S_i$, contradicting the assumption of the present case. Hence for all i we have $V(K) \cap S_i \neq \emptyset$. We know that $|V(K) \cap S_1| \leq \lfloor \frac{|K|}{2} \rfloor$. Hence by Eq. (8) $|T| + |V(K) \cap S_1| < \frac{|V|}{2}$, if $x \in S_1$ is isolated in $L - T$, then there exists another isolated vertex $y \in S_1$ such that $(x, y) \in E(G)$. If there is an edge (a, b) such that $a \in V(K) \cap S_2$ and $b \in V(K) \cap S_3$, then the cycle (x, a, y, b) in $\bar{G}$ is chordless, contradicting the assumption that $\bar{G}$ is chordal. Thus there is no edge between any two vertices $a \in V(K) \cap S_2$ and $b \in V(K) \cap S_3$. Hence K is bipartite with parts $V(K) \cap S_1$ and $V(K) \setminus S_1$. Thus K is necessarily an even component, and $|V(K) \cap S_1| = \frac{|K|}{2}$. Hence $|S_1| \geq |\mathcal{O}| + \frac{|K|}{2} > |T| + \frac{|K|}{2} = |T| + |V(K) \setminus S_1| \geq |T_2| + |T_3| + |V(K) \setminus S_1| = |S_2| + |S_3|$. This contradicts the assumption that $|S_1| \leq \frac{|V|}{2}$. □

We can now prove Theorem 4.1:

Proof. Let $S_1, S_2, \ldots, S_t$ be the parallelism classes of $\mathcal{M}$. Since $\mathcal{M}$ is of rank 2, $\Delta(\mathcal{M}) = \max(\max |S_i|, \frac{|V|}{2})$. As in the proof of Theorem 3.1, we may assume that $\chi(\mathcal{M}) = \Delta(M)$ and $\Delta(M) \geq D(G) + 1$.

Assume first that $\Delta(\mathcal{M}) = \max |S_i|$. Without loss of generality $\Delta(\mathcal{M}) = |S_1|$. Let $V_1 = S_1$ and $V_2 = V \setminus S_1$. Since G is chordal and $D(G) \leq \Delta(\mathcal{M}) - 1 = |V_1| - 1$, by Lemma 4.2 there exists a matching F of V_2 into V_1 in $\bar{G} \cap \mathcal{M}$. Let A be the set of those vertices in S_1 that are not saturated by F. Then $F \cup (\bigcup_{v \in A}\{v\})$ is a partition of $V(G)$ into sets belonging to $\mathcal{I}(G) \cap \mathcal{M}$, and its size is $|S_1|$. Thus, $\chi(\mathcal{M} \cap \mathcal{I}(G)) \leq |S_1| = \Delta(\mathcal{M})$, and the theorem is proved.

Assume next that $\Delta(\mathcal{M}) = \frac{|V|}{2}$. Then $|S_i| \leq \frac{|V|}{2}$ for all i. We have $d(\bar{G}) = |V| - 1 - D(G) \geq |V| - 1 - (\Delta(\mathcal{M}) - 1) = \frac{|V|}{2}$. By Lemma 4.3 (applied to $\bar{G}$), $\bar{G} \cap \mathcal{M}$ has a perfect matching, which is a partition of $V(G)$ into $\frac{|V|}{2}$ two-way independent sets, proving the desired result $\chi(\mathcal{M} \cap \mathcal{I}(G)) \leq \frac{|V|}{2}$. □

5. A dual conjecture

Conjectures 2.4 and 2.5 have dual counterparts, on packing bases, rather than covering by independent sets. To formulate them, we shall need some notation. As usual, for a set X we denote by $\mathcal{M}.X$ the contraction of $\mathcal{M}$ to X, namely the matroid defined by: $J \in \mathcal{M}.X$ if $I \cup J \in \mathcal{M}$ for all sets I in $\mathcal{M}$ contained in $V \setminus X$.

Definition 5.1. Given a matroid $\mathcal{M}$, we write $\delta(\mathcal{M})$ for $\min \frac{|X|}{\rho(\mathcal{M}.X)}$, where the minimum ranges over all subsets X of $V = V(\mathcal{M})$. By $\pi(\mathcal{M})$ we denote the packing number of the bases of $\mathcal{M}$, namely the maximal number of disjoint bases of $\mathcal{M}$.

The source of the notation $\Delta(\mathcal{M})$ and $\delta(\mathcal{M})$ is in that in a partition matroid these are respectively the largest and smallest sizes of parts of the matroid. Thus in the matroid $\mathcal{M}$ induced on the edge set of a bipartite graph by one side of the graph, $\Delta(\mathcal{M})$ and $\delta(\mathcal{M})$ are the largest and smallest degrees, respectively.

Definition 5.2. Given a simplicial complex $\mathcal{C}$ and a matroid $\mathcal{M}$ on the same ground set V, an $[\mathcal{M}, \mathcal{C}]$-*matching* is a base of $\mathcal{M}$ belonging to $\mathcal{C}$. If such a matching exists, we say that the pair $[\mathcal{M}, \mathcal{C}]$ is *matchable*.

A dual theorem to Theorem 1.3, also proved by Edmonds [7], is:

Theorem 5.3. $\pi(\mathcal{M}) = \lfloor \delta(\mathcal{M}) \rfloor$.

In [1] the following theorem was proved:

Theorem 5.4. *Let $\mathcal{M}$ and G be a matroid and a graph on the same vertex set. If $\pi(\mathcal{M}) \geq 2D(G)$ then the pair $[\mathcal{M}, \mathcal{I}(G)]$ is matchable.*

For chordal graphs the condition can be weakened:

Theorem 5.5. *If G is chordal and $\delta(\mathcal{M}) \geq D(G) + 1$ then the pair $[\mathcal{M}, \mathcal{I}(G)]$ is matchable.*

We conjecture the following:

Conjecture 5.6. *Let $\mathcal{M}$ be a truncated partition matroid, and let G be a graph on the same vertex set. If $\pi(\mathcal{M}) \geq 2D(G)$ then there exist $\pi(\mathcal{M})$ disjoint bases of $\mathcal{M}$ that are independent in G. If $\mathcal{M}$ is a truncation of a partition matroid and G is chordal and $\pi(\mathcal{M}) \geq D(G) + 1$ then there exist $\pi(\mathcal{M})$ disjoint bases of $\mathcal{M}$ that are independent in G.*

The methods used in the previous sections prove this conjecture for matroids of rank 2.

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